

# Translation lengths in $Out(F_n)$

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## Abstract

We prove that all elements of infinite order in  $Out(F_n)$  have positive translation lengths; moreover, they are bounded away from zero. Consequences include a new proof that solvable subgroups of  $Out(F_n)$  are finitely generated and virtually abelian and the new result that such subgroups are quasi-convex.

## 1 Introduction

In this paper we will study the translation lengths of outer automorphisms of a free group. Following [GS91] we define the translation length  $\tau_{X,G}(g)$  of  $g \in \Gamma$  to be

$$\lim_{n \rightarrow \infty} \frac{\|g^n\|}{n}$$

where  $\Gamma$  is a group with finite generating set  $X$ , and  $\|g\|$  denotes the length of  $g$  in the word metric on  $\Gamma$  associated to  $X$ .

Farb, Lubotzky and Minsky proved that Dehn twists (more generally, all elements of infinite order) in  $Mod(\Sigma_g)$  have positive translation length ([FLM]). We prove

**Theorem 1.1.** *Every infinite order element  $\mathcal{O} \in Out(F_n)$  has positive translation length. Furthermore, there exists a positive constant  $\varepsilon_n$  such that  $\tau(\mathcal{O}) \geq \varepsilon_n$ .*

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Once more we can see the strong analogy between mapping class group of a surface,  $Mod(\Sigma_g)$ , and outer automorphism group of a free group,  $Out(F_n)$ .

To prove their theorem, Farb, Lubotzky and Minsky found a way to measure how much a Dehn twist is ‘twisted’ by looking at simple closed curves and their intersection number. Such an approach cannot work in the case of  $Out(F_n)$  as we do not have an analogue of the intersection number.

As a consequence of our main result we have

**Corollary 1.2.** *Every solvable subgroup of  $Out(F_n)$  is finitely generated and virtually abelian.*

Corollary 1.2 was proved in [BFH99a], but Theorem 1.1 offers an alternative proof.

**Corollary 1.3.** *Every abelian subgroup  $A$  of  $Out(F_n)$  is quasi-convex.*

The proofs use techniques of [GS91] and follow the same lines as the corresponding proofs in [Bes99].

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## 2 Translation lengths

From the definition of translation length we can see that it depends on the choice of generating set for a group  $\Gamma$ . We will omit the reference to the generating set, since it will be clear which one we are using.

We list some properties of translation lengths which can be found in [GS91].

**Proposition 2.1.** *Let  $X$  be a generating set for a group  $\Gamma$ .*

1.  $0 \leq \tau(g) \leq \|g\|$
2. For all  $x, g \in G$ ,  $\tau(xgx^{-1}) = \tau(g)$ .
3.  $\tau(g^n) = n \cdot \tau(g) \ \forall n \in \mathbb{N}$ .

Let  $X = \{x_1, x_2, \dots, x_n\}$  be a set of generators of a free group  $F_n$ . Let  $Y$  be the set of generators for  $Aut(F_n)$  consisting of:

1. permutations  $(x_i \mapsto x_j, x_j \mapsto x_i, x_k \mapsto x_k \text{ for all } k \neq i, j)$ ,
2. inversions  $(x_i \mapsto x_i^{-1}, x_j \mapsto x_j \text{ for all } j \neq i)$ ,
3. Dehn twists  $(x_i \mapsto x_i x_j, x_k \mapsto x_k \text{ for all } k \neq i)$ .

Let  $\tilde{Y}$  denote the generating set for  $Out(F_n)$  consisting of equivalence classes of elements of  $Y$ .

Our goal is to prove that every element of infinite order in  $Out(F_n)$  has positive translation length. Since  $Aut(F_n)$  embeds into  $Out(F_{n+1})$ , it will follow that every infinite order element of  $Aut(F_n)$  has positive translation length.

We will need the following definition for our proof:

**Definition 2.2.** Define a map  $\alpha : F_n \rightarrow \mathbb{N}$  by

$$\alpha(w) = \max\{|p| : \tilde{w}^p \text{ is a subword of } w\},$$

where elements of  $F_n$  are regarded as reduced words in the generators and their inverses. We also define

$$\tilde{\alpha}([w]) = \max\{\alpha(u) : u \text{ is a cyclically reduced conjugate of } w\}$$

for the conjugacy class,  $[w]$ , of  $w$ .

**Lemma 2.3.** *There exists a constant  $C > 0$  such that for any  $\tilde{g} \in \tilde{Y}$  and any cyclically reduced word  $w \in F_n$  we have*

$$\tilde{\alpha}(\tilde{g}([w])) \leq \tilde{\alpha}([w]) + C.$$

*Proof.* Note that inversions and permutations do not affect  $\alpha(w)$ , so we need only consider the case where  $g$  is a Dehn twist.

Let  $w \in F_n$  be a cyclically reduced element of length  $n$ . Let  $w = A \tilde{w}^p B$ , where  $\alpha(w) = |p|$ . Consider

$$g(w) = [[g(A)]] [[g(\tilde{w})^p]] [[g(B)]],$$

where  $[[g(w)^p]]$  denotes the reduced word obtained from  $g(w)^p$ . By the *Bounded Cancellation Lemma* ([Coo87]) there is a constant  $C(g)$  such that at most  $C(g)$  cancellations occur after concatenation of the words  $[[g(A)]]$  and  $[[g(\tilde{w})^p]]$ . Hence  $p$  can decrease by at most  $2C(g)$  (cancellations may occur at

the beginning and at the end of  $[[g(\tilde{w})^p]]$ . Let  $C_g = 2 \max\{C(g), C(g^{-1})\}$ . We now have

$$\begin{aligned}\alpha([g(w)]) &\geq \alpha(w) - C_g \\ \alpha(w) = \alpha(g^{-1}(g(w))) &\geq \alpha([g(w)]) - C_g \\ \alpha([g(w)]) &\leq \alpha(w) + C_g.\end{aligned}$$

If we take  $\tilde{C} = \max\{C_g : g \text{ a Dehn twist in } Y\}$ , our claim is proved for elements of  $Y$ . Using a similar argument, we see that there is a constant  $C$  such that

$$\tilde{\alpha}(\tilde{g}([w])) \leq \tilde{\alpha}([w]) + C.$$

□

**Example 2.4.** We illustrate the idea of the proof of Theorem 1.1 with an example of a Dehn twist. Let  $g$  be a Dehn twist which sends  $x_2$  to  $x_2x_1$  and fixes all other generators of  $F_n$ .

$$\alpha(g^n(x_2)) = \alpha(x_2x_1^n) = n.$$

If  $g^n = g_1 \cdots g_m$ , then  $\|g^n\| = m$ . By Lemma 2.3, we have that

$$\begin{aligned}n = \alpha(g^n(x_2)) &\leq \alpha(x_2) + mC = mC + 1, \\ \tau(g) = \lim_{n \rightarrow \infty} \frac{\|g^n\|}{n} &\geq \lim_{n \rightarrow \infty} \frac{n-1}{nC} = \frac{1}{C} > 0.\end{aligned}$$

So  $g$  has positive translation length.

We give a short list of definitions which will be used throughout the rest of the paper, but we suggest that the reader look at [BFH99b].

Every element  $\mathcal{O} \in \text{Out}(F_n)$  can be represented by a homotopy equivalence  $f: G \rightarrow G$  of a graph  $G$  whose fundamental group is identified with  $F_n$ . A map  $\sigma: J \rightarrow G$  ( $J$  is an interval) is called a *path* if it is either locally injective or a constant map (we also require that the endpoints of  $\sigma$  are at vertices). Every map  $\sigma: J \rightarrow G$  is homotopic (relative endpoints) to a path  $[[\sigma]]$ .

If  $\sigma = \sigma_1 \dots \sigma_l$  is a decomposition of a path or a circuit  $\sigma$  into nontrivial subpaths we say that it is a *k-splitting* if

$$f^k(\sigma) = [[f^k(\sigma_1)]] \dots [[f^k(\sigma_l)]]$$

is a decomposition into subpaths and is a *splitting* if it is a  $k$ -splitting for all  $k > 0$ .

We say that a nontrivial path  $\sigma \in G$  is a *Nielsen path* for  $f : G \rightarrow G$  if  $[[f(\sigma)]] = \sigma$ . The Nielsen path  $\sigma$  is *indivisible* if it cannot be written as a concatenation of nontrivial Nielsen paths.

Let  $= G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_K = G$  be a filtration of  $G$  by  $f$ -invariant subgraphs, and let  $H_i = G_i \setminus G_{i-1}$ . Suppose  $H_i$  is a single edge  $E_i$  and  $f(E_i) = E_i v^l$  for some closed indivisible Nielsen path  $v \subset G_{i-1}$  and some  $l > 0$ . The *exceptional paths* are paths of the form  $E_i v^k \overline{E_j}$  or  $E_i \overline{v^k} \overline{E_j}$ , where  $k \geq 0$ ,  $j \leq i$  and  $f(E_j) = E_j v^m$ , for  $m > 0$ .

We remind the reader that every element of  $Out(F_n)$  of infinite order has either exponential or polynomial growth ([BH92]). A polynomially growing outer automorphism  $\mathcal{O} \in Out(F_n)$  is unipotent if its action in  $H_1(F_n; \mathbb{Z})$  is unipotent (UPG automorphism).

The following Theorem can be found in [BFH00](page 564).

**Theorem 2.5.** *Suppose that  $\mathcal{O} \in Out(F_n)$  is a UPG automorphism. Then there is a topological representative  $f : G \rightarrow G$  of  $\mathcal{O}$  with the following properties:*

1. *Each  $G_i$  is the union of  $G_{i-1}$  and a single edge  $E_i$  satisfying  $f(E_i) = E_i \cdot u_i$  for some closed path  $u_i$  that crosses only edges in  $G_{i-1}$  ( $\cdot$  indicates that the decomposition in question is a splitting).*
2. *If  $\sigma$  is any path with endpoints at vertices, then there exists  $M = M(\sigma)$  so that for each  $m \geq M$ ,  $[[f^m(\sigma)]]$  splits into subpaths that are either single edges or exceptional subpaths.*

□

**Lemma 2.6.** *Let  $\mathcal{O} \in Out(F_n)$  be a UPG automorphism of infinite order and let  $f : G \rightarrow G$  be its topological representative as in Theorem 2.5. For every path  $\gamma$  in  $G$  for which  $[[f(\gamma)]] \neq \gamma$  there exists  $a \in \mathbb{R}$  such that*

$$\alpha([[f^k(\gamma)]]) \geq k + a.$$

*Proof.* We prove our claim by induction on the (minimal) index,  $m$ , of the filtration element that contains a path  $\gamma$ .

If  $\gamma \subset G_1$  there is nothing to be proved since  $G_1$  contains only one edge  $E_1$  which is fixed by  $f$ .

Suppose the claim is true for the subpaths contained in  $G_{m-1}$  that satisfy out hypothesis, and let  $\gamma$  be a path in  $G_m$  for which  $[[f(\gamma)]] \neq \gamma$ . By Theorem 2.5 for every  $m \geq M(\gamma)$ ,  $[[f^m(\gamma)]]$  splits into subpaths that are either single edges or exceptional paths. Denote  $[[f^{M(\gamma)}(\gamma)]]$  by  $\tilde{\gamma}$ , so that  $\tilde{\gamma} = \gamma_1 \cdot \dots \cdot \gamma_p$ , where  $\gamma_i$  is either a single edge or an exceptional path.

Assume there is an exceptional path  $\gamma_t$  which is not fixed by  $f$ . Without loss of generality we may assume that  $\gamma_t = E_i v^r \overline{E_j}$ , where  $f(E_i) = E_i v^l$  ( $l > 0$ ),  $f(E_j) = E_j v^s$  ( $s > 0$ ) and  $j \leq i$ . Now we have that

$$[[f^k(\gamma_t)]] = E_i v^{k(l-s)+r} \overline{E_j},$$

and

$$\alpha([[f^k(\gamma_t)]]) \geq k(l-s) + r, \quad \text{if } l-s > 0,$$

$$\alpha([[f^k(\gamma_t)]]) \geq k(s-l) - r, \quad \text{if } l-s < 0.$$

Since  $\gamma_t$  is not fixed,  $l$  and  $s$  cannot be equal. Therefore

$$\alpha([[f^k(\tilde{\gamma})]]) \geq k \pm r,$$

$$\alpha([[f^k(\gamma)]]) = \alpha([[f^{k-M(\gamma)}(\tilde{\gamma})]]) \geq k - M(\gamma) \pm r.$$

If all exceptional paths in  $\tilde{\gamma}$  are fixed, there exists an edge  $\gamma_t = E_i$  which is not fixed by  $f$ . We know that  $f(E_i) = E_i \cdot u_i$ , where  $u_i$  is a closed path contained in  $G_{m-1}$ .

If  $[[f(u_i)]] = u_i$ , our claim is proven since  $[[f^k(E_i)]] = E_i u_i^k$  and so

$$\alpha([[f^k(\tilde{\gamma})]]) \geq k,$$

$$\alpha([[f^k(\gamma)]]) \geq k - M(\gamma).$$

If  $[[f(u_i)]] \neq u_i$ , there exists  $a \in \mathbb{R}$  such that  $\alpha([[f^k(u_i)]]) \geq k + a$ . We now have

$$\alpha([[f^k(\gamma)]]) = \alpha([[f^{k-M(\gamma)}(\tilde{\gamma})]]) \geq \alpha([[f^{k-M(\gamma)}(u_i)]]) \geq k - M(\gamma) + a.$$

□

**Lemma 2.7.** *Let  $\mathcal{O}$  be a UPG automorphism of  $F_n$  of infinite order. There exist a closed path  $\sigma$  in  $G$ , and  $b \in \mathbb{R}$  such that*

$$\tilde{\alpha}(\mathcal{O}^k(\sigma)) \geq k + b.$$

*Proof.* Let  $f: G \rightarrow G$  be as in Theorem 2.5. Since  $\mathcal{O} \neq id$  there is a closed path  $\sigma$  which is not fixed by  $f$ . We know that for every  $m \geq M(\sigma)$ ,  $[[f^m(\sigma)]] = \sigma_1 \cdot \dots \cdot \sigma_p$  splits into subpaths that are either single edges or exceptional paths. Denote  $[[f^{M(\sigma)}(\sigma)]]$  by  $\tilde{\sigma}$ , so that  $\tilde{\sigma} = \sigma_1 \cdot \dots \cdot \sigma_p$ .

If there is an exceptional path  $\sigma_t$  in this splitting which is not fixed by  $f$ , we get

$$\tilde{\alpha}(\mathcal{O}^k(\tilde{\sigma})) \geq k \pm r$$

as in Lemma 2.6.

If all exceptional paths in  $\tilde{\sigma}$  are fixed, there exists an edge  $\sigma_t = E_i$  such that  $f(E_i) = E_i \cdot u_i$ , where  $u_i$  is a closed path contained in  $G_{i-1}$ . By Lemma 2.6 there exists  $a \in \mathbb{R}$  such that

$$\alpha(f^k(E_i)) \geq k + a.$$

Hence, in all the above cases, there is  $b \in \mathbb{R}$  such that

$$\tilde{\alpha}(\mathcal{O}^k(\tilde{\sigma})) \geq k + b.$$

□

### 3 Proof of Theorem 1.1

We consider the cases of exponentially and polynomially growing outer automorphisms separately.

**Case 1.** Let  $\mathcal{O}$  be an exponentially growing outer automorphism of  $F_n$ . There exist  $\lambda > 1$  and a cyclically reduced word  $w$  such that  $\ell(\mathcal{O}^k([w])) \geq \lambda^k \ell([w])$ , for all  $k \geq 1$ , where  $\ell$  denotes the cyclic word length (see [BH92]). Suppose that  $\mathcal{O}^k$  can be written as  $\tilde{g}_1 \dots \tilde{g}_m$ , for some  $\tilde{g}_i \in \tilde{Y}$ . It is straightforward to show that for all  $\tilde{g} \in \tilde{Y}$  and any cyclically reduced word  $w$  we have

$$\ell(\tilde{g}([w])) \leq 2 \ell([w])$$

Using this inequality we obtain:

$$\lambda^k \ell([w]) \leq \ell(\mathcal{O}^k([w])) \leq 2^m \ell([w])$$

Hence

$$m \geq \frac{\log \lambda^k}{\log 2}$$

which implies

$$\tau(\mathcal{O}) \geq \frac{\log \lambda}{\log 2} > 0$$

There is a constant  $c_1 > 1$  such that  $\lambda \geq c_1$  ([BH92]). Therefore  $\tau(\mathcal{O})$  is bounded away from zero.

**Case 2.** Let  $\mathcal{O}$  be a *UPG* automorphism. Again assume that  $\mathcal{O}^k$  can be written as  $\tilde{g}_1 \dots \tilde{g}_m$ , for some  $\tilde{g}_i \in \tilde{Y}$ . By Lemma 2.7 there is a closed path  $\sigma$  in  $G$  such that

$$\tilde{\alpha}(\mathcal{O}^k(\sigma)) \geq k + b$$

Let  $u_j = \tilde{g}_j \dots \tilde{g}_m$ . Applying Lemma 2.3 we get

$$\tilde{\alpha}(u_i(\sigma)) \leq \tilde{\alpha}(u_{i+1}(\sigma)) + C$$

which yields

$$\begin{aligned} k + b &\leq \tilde{\alpha}(\mathcal{O}^k(\sigma)) \leq mC + \tilde{\alpha}(\sigma) \\ \frac{k + b - \tilde{\alpha}(\sigma)}{C} &\leq m. \end{aligned}$$

We have

$$\tau(\mathcal{O}) \geq \lim_{k \rightarrow \infty} \frac{k + b - \tilde{\alpha}(\sigma)}{kC} = \frac{1}{C}.$$

Finally, if  $\mathcal{O}$  is any polynomially growing outer automorphism, then there exists  $s \geq 1$  (bounded above by some  $c_2$ ), such that  $\mathcal{O}^s$  is a *UPG* automorphism. Then

$$\tau(\mathcal{O}) = \frac{1}{s} \tau(\mathcal{O}^s) \geq \frac{1}{Cs} > 0.$$

Since  $s$  is bounded by  $c_2$ , we get  $\tau(\mathcal{O}) \geq \frac{1}{Cc_2} > 0$ .

This completes the proof.  $\square$



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